

GENERALIZED STIRLING TRANSFORM

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ABSTRACT. In this paper, algorithms are developed for computing the Stirling transform and the inverse Stirling transform; specifically, we investigate a class of sequences satisfying a two-term recurrence. We derive a general identity which generalizes the usual Stirling transform and investigate the corresponding generating functions also. In addition, some interesting consequences of these results related to classical sequences like Fibonacci, Bernoulli and the numbers of derangements have been derived.

1. INTRODUCTION

The Stirling numbers arise frequently in mathematics, especially in enumerative problems. This is the reason of their important role in combinatorial analysis, number theory, probability, graph theory, calculus of finite differences and interpolation. The notations for these numbers have never been standardized, this paper follows the notation of Riordan for the signed Stirling numbers of the first kind $s(n, k)$ and Knuth's notation for the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

The Stirling transform of a sequence (a_n) is the the sequence (b_n) given by

$$(1.1) \quad b_n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} a_k,$$

and the inverse transform is

$$(1.2) \quad a_n = \sum_{k=0}^n s(n, k) b_k.$$

The identity (1.1) has a combinatorial interpretation given in [2]. If a_n is the number of objects in some class with points labeled $1, 2, \dots, n$ (with all labels distinct) then b_n is the number of objects with points labeled $1, 2, \dots, n$ (with repetitions allowed).

In this paper, algorithms are developed for computing the Stirling transform and the inverse Stirling transform; specifically, we investigate a class of sequences satisfying a two-term recurrence. We derive a general identity

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which generalizes the usual Stirling transform and investigate the corresponding generating functions also.

Given a sequence $a_m := a_{0,m}$ ($m \geq 0$). We construct an infinite matrix $\mathcal{S} := (a_{n,m})$ as follows:

The first row $a_{0,m}$ of the matrix is the initial sequence; the first column $b_n := a_{n,0}$ ($n \geq 0$) is called the final sequence and, each entry $a_{n,m}$ is given recursively by

$$(1.3) \quad a_{n+1,m} = a_{n,m+1} + m a_{n,m}.$$

Conversely, if we start with the final sequence, the matrix \mathcal{S} can be recovered by the recursive relations

$$(1.4) \quad a_{n,m+1} = a_{n+1,m} - m a_{n,m}.$$

2. DEFINITIONS AND NOTATION

In this section, we introduce some definitions and notations which are useful in the rest of the paper. \mathbb{N} being the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The falling and rising factorials are defined, respectively by

$$(x)_n = x(x-1) \cdots (x-n+1), (x)_0 = 1$$

and

$$\langle x \rangle_n = x(x+1) \cdots (x+n-1), \langle x \rangle_0 = 1.$$

The (signed) Stirling numbers $s(n, k)$ of the first kind, which are usually defined by

$$(2.1) \quad (x)_n = \sum_{k=0}^n s(n, k) x^k,$$

or by the following generating function

$$(2.2) \quad \frac{1}{k!} (\ln(1+x))^k = \sum_{n \geq k} s(n, k) \frac{x^n}{n!}.$$

It follows from (2.1) or (2.2) that

$$(2.3) \quad s(n+1, k) = s(n, k-1) - n s(n, k)$$

and that

$$s(n, 0) = \delta_{n,0} \quad (n \in \mathbb{N}), \quad s(n, k) = 0 \quad (k > n \text{ or } k < 0),$$

where $\delta_{n,m}$ denotes the Kronecker symbol.

The Stirling numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ of the second kind count the number of possible partitions of a set of n objects into k disjoint blocks. These numbers can be defined explicitly by

$$x^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (x)_k.$$

For any positive $r \in \mathbb{N}$ the quantity $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ denotes the number of partitions of a set of n objects into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. These numbers obey the recurrence relation

$$(2.4) \quad \begin{cases} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r = 0, & n < r, \\ \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r = \delta_{k,r}, & n = r, \\ \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}_r, & n > r, \end{cases}$$

The exponential generating function is given by

$$(2.5) \quad \sum_{n \geq k} \left\{ \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\}_r \frac{x^n}{n!} = \frac{1}{k!} e^{rx} (e^x - 1)^k.$$

The properties

$$\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}_r = r^{n-r}$$

and

$$(2.6) \quad \left\{ \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\}_r = \left\{ \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\}_{r-1} - (r-1) \left\{ \begin{smallmatrix} n+r-1 \\ k+r \end{smallmatrix} \right\}_{r-1}$$

are given in [3], which one can consult for more details on r -Stirling numbers.

3. COMBINATORIAL IDENTITIES

Theorem 1. *Given an initial sequence $(a_{0,m})_{m \geq 0}$, define the matrix \mathcal{S} by (1.3). Then, the entries of the infinite matrix \mathcal{S} are given by*

$$(3.1) \quad a_{n,m} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n+m \\ k+m \end{smallmatrix} \right\}_m a_{0,m+k}.$$

Proof. We proof by induction on n , the result clearly holds for $n = 0$. By induction hypothesis

$$\begin{aligned}
a_{n,m+1} + ma_{n,m} &= \sum_{k=0}^n \left\{ \begin{matrix} n+m+1 \\ k+m+1 \end{matrix} \right\}_{m+1} a_{0,m+k+1} + m \left\{ \begin{matrix} n+m \\ m \end{matrix} \right\}_m a_{0,m} \\
&\quad + m \sum_{k=1}^{n-1} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m a_{0,m+k} \\
&= \sum_{k=0}^n \left\{ \begin{matrix} n+m+1 \\ k+m+1 \end{matrix} \right\}_{m+1} a_{0,m+k+1} + m \left\{ \begin{matrix} n+m \\ m \end{matrix} \right\}_m a_{0,m} \\
&\quad + m \sum_{k=0}^{n-2} \left\{ \begin{matrix} n+m \\ k+m+1 \end{matrix} \right\}_m a_{0,m+k+1} \\
&= \left\{ \begin{matrix} n+m+1 \\ n+m+1 \end{matrix} \right\}_{m+1} a_{0,m+n+1} + \left\{ \begin{matrix} n+m+1 \\ n+m \end{matrix} \right\}_{m+1} a_{0,m+n} \\
&\quad + \sum_{k=0}^{n-2} \left\{ \begin{matrix} n+m+1 \\ k+m+1 \end{matrix} \right\}_{m+1} a_{0,m+k+1} \\
&\quad + m \left\{ \begin{matrix} n+m \\ m \end{matrix} \right\}_m a_{0,m} + m \sum_{k=0}^{n-2} \left\{ \begin{matrix} n+m \\ k+m+1 \end{matrix} \right\}_m a_{0,m+k+1}.
\end{aligned}$$

From (2.6) and after some rearrangements, we get

$$\begin{aligned}
a_{n,m+1} + ma_{n,m} &= \sum_{k=0}^{n+1} \left\{ \begin{matrix} n+m+1 \\ k+m \end{matrix} \right\}_m a_{0,m+k} \\
&= a_{n+1,m}.
\end{aligned}$$

□

Theorem 2. Given a final sequence $(a_{n,0})_{n \geq 0}$, define the matrix \mathcal{S} by (1.4). Then, the entries of the infinite matrix \mathcal{S} are given by

$$(3.2) \quad a_{n,m} = \sum_{k=0}^m s(m, k) a_{n+k,0}.$$

Proof. We proof by induction on m , the result clearly holds for $n = 0$. By induction hypothesis and (2.3), we have

$$\begin{aligned}
a_{n+1,m} - ma_{n,m} &= \sum_{k=1}^{m+1} s(m, k-1) a_{n+k,0} - m \sum_{k=0}^m s(m, k) a_{n+k,0} \\
&= s(m, m) a_{n+m+1,0} + \sum_{k=1}^m s(m, k-1) a_{n+k,0} \\
&\quad - ms(m, 0) a_{n,0} - m \sum_{k=1}^m s(m, k) a_{n+k,0} \\
&= s(m, m) a_{n+m+1,0} + \sum_{k=1}^m (s(m, k-1) - ms(m, k)) a_{n+k,0} \\
&\quad - ms(m, 0) a_{n,0} \\
&= a_{n,m+1}.
\end{aligned}$$

□

Corollary 1.

$$(3.3) \quad \sum_{k=0}^m s(m, k) b_{n+k} = \sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m a_{m+k}.$$

The last identity can be viewed as the generalized Stirling transform which reduced, for $m = 0$, to the Stirling transform (1.1) of the sequence a_n , and for $n = 0$ reduces to the inverse Stirling transform (1.2) of the sequence b_m .

We may now formulate the following algorithms

Algorithm 1. Stirling transform

Input: a_n

Output: b_n

Set $X_m = a_{n-m,m}$

for $n = 0, 1, \dots$ **do**

$X_n := a_n$

for $m = n, n-1, \dots, 0$ **do**

$X_{m-1} := (m-1) X_{m-1} + X_m$

end do

$b_n := X_0$

end do

Algorithm 2. inverse Stirling transform

Input: b_m

Output: a_m

Set $Y_n = b_{n,m-n}$

for $m = 0, 1, \dots$ **do**

$Y_m := b_m$

for $n = m, m-1, \dots, 0$ **do**

$Y_{n-1} := Y_n - (m - n) Y_{n-1}$
end do
 $a_m := Y_0$
end do

Example 1. Setting $a_{0,m} = 1$ in (3.3), we get the well known identity [8]

$$\sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m = \sum_{k=0}^m s(m, k) B_{n+k},$$

where B_n is the n th Bell number.

Example 2. Let $(F_n)_{n \in \mathbb{N}_0}$ be the Fibonacci sequence given by Binet's formula

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. If the initial sequence $a_{0,m} = \frac{(-1)^m}{\sqrt{5}} (\langle -\alpha \rangle_m - \langle -\beta \rangle_m)$, then we get the following matrix

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & 0 & 1 & -4 & 19 & -108 & \cdots \\ 1 & 1 & 1 & -1 & 3 & -13 & 71 & \\ 1 & 2 & 1 & 0 & -1 & 6 & -37 & \\ 2 & 3 & 2 & -1 & 2 & -7 & 34 & \\ 3 & 5 & 3 & -1 & 1 & -1 & -3 & \\ 5 & 8 & 5 & -2 & 3 & -8 & 31 & \\ 8 & 13 & 8 & -3 & 4 & -9 & 28 & \\ 13 & 21 & 13 & -5 & 7 & -17 & 59 & \\ \vdots & & & & & & & \end{pmatrix}.$$

From this matrix we observe that $a_{n,0} = a_{n,2} = -a_{n+2,3} = F_n$, and $a_{n+3,4} = L_n$, where $(L_n)_{n \in \mathbb{N}_0}$ the Lucas sequence given by Binet's formula

$$L_n = \alpha^n + \beta^n.$$

It is well known that the F_n and L_n are connected by the formula

$$L_n = F_{n-1} + F_{n+1}, \quad (n \in \mathbb{N}).$$

By (3.3), one can deduce that

$$\sum_{k=0}^m s(m, k) F_{n+k} = \frac{1}{\sqrt{5}} \sum_{k=0}^n (-1)^{m+k} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m (\langle -\alpha \rangle_{m+k} - \langle -\beta \rangle_{m+k}),$$

and by Theorem 2, we get

$$\begin{aligned} F_n &= \sum_{k=0}^2 s(2, k) F_{n+k} = -F_{n+1} + F_{n+2} \\ &= -\sum_{k=0}^3 s(3, k) F_{n+2+k} = -2F_{n+3} + 3F_{n+4} - F_{n+5}, \end{aligned}$$

and for $n \in \mathbb{N}_0$

$$L_n = \sum_{k=0}^4 s(4, k) F_{n+3+k} = -6F_{n+4} + 11F_{n+5} - 6F_{n+6} + F_{n+7}.$$

By Theorem 1

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (\langle -\alpha \rangle_k - \langle -\beta \rangle_k) \\ &= \frac{1}{\sqrt{5}} \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n+2 \\ k+2 \end{matrix} \right\}_2 (\langle -\alpha \rangle_{k+2} - \langle -\beta \rangle_{k+2}) \\ &= \frac{1}{\sqrt{5}} \sum_{k=0}^{n+2} (-1)^k \left\{ \begin{matrix} n+5 \\ k+3 \end{matrix} \right\}_3 (\langle -\alpha \rangle_{k+3} - \langle -\beta \rangle_{k+3}), \end{aligned}$$

and

$$L_n = \frac{1}{\sqrt{5}} \sum_{k=0}^{n+3} (-1)^k \left\{ \begin{matrix} n+7 \\ k+4 \end{matrix} \right\}_4 (\langle -\alpha \rangle_{k+4} - \langle -\beta \rangle_{k+4}).$$

4. GENERATING FUNCTION

Theorem 3. Suppose that the initial sequence $a_{0,m+r}$ has the following exponential generating function $A_r(z) = \sum_{k \geq 0} a_{0,k+r} \frac{z^k}{k!}$. Then the sequence $\{a_{n,r}\}_n$

of the r th columns of the matrix \mathcal{S} has an exponential generating function $\mathcal{B}_r(z) = \sum_{n \geq 0} a_{n,r} \frac{z^n}{n!}$ given by

$$(4.1) \quad B_r(z) = e^{rz} A_r(e^z - 1)$$

Proof. We have

$$\begin{aligned} B_r(z) &= \sum_{k \geq 0} a_{0,r+k} \sum_{n \geq 0} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \frac{z^n}{n!} \\ &= \sum_{k \geq 0} a_{0,r+k} \frac{1}{k!} e^{rz} (e^z - 1)^k \\ &= e^{rz} \sum_{k \geq 0} a_{0,r+k} \frac{(e^z - 1)^k}{k!} \\ &= e^{rz} A_r(e^z - 1). \end{aligned}$$

□

Theorem 4. Suppose that the final sequence $a_{n+r,0}$ has the following exponential generating function $\mathcal{B}_r(z) = \sum_{k \geq 0} a_{k+r,0} \frac{z^k}{k!}$. Then the sequence $\{a_{r,m}\}_m$

of the r th rows of the matrix \mathcal{S} has an exponential generating function

$$\mathcal{A}_r(z) = \sum_{m \geq 0} a_{r,m} \frac{z^m}{m!} \text{ given by}$$

$$(4.2) \quad \mathcal{A}_r(z) = \mathcal{B}_r(\ln(1+z)).$$

Proof. We have

$$\begin{aligned} \mathcal{A}_r(z) &= \sum_{k \geq 0} a_{r+k,0} \sum_{m \geq 0} s(m, k) \frac{z^m}{m!} \\ &= \sum_{k \geq 0} a_{r+k,0} \frac{(\ln(1+z))^k}{k!} \\ &= \mathcal{B}_r(\ln(1+z)). \end{aligned}$$

□

Example 3. A derangement on a set $\{1, 2, \dots, m\}$ is a permutation $\pi = i_1 i_2 \dots i_m$ such that $i_k \neq k$ for $k = 1, 2, \dots, m$. The number of derangements on $\{1, 2, \dots, m\}$ is denoted by D_m and given by $D_m = \left[\frac{m!}{e} \right]$, where $[x]$ the nearest integer function. Now, if the initial sequence $a_{0,m} = (-1)^m D_m$, then we get the following matrix

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 1 & -2 & 9 & -44 & \dots \\ 0 & 1 & 0 & 3 & -8 & 45 & \\ 1 & 1 & 3 & 1 & 13 & -39 & \\ 1 & 4 & 7 & 16 & 13 & 76 & \\ 4 & 11 & 30 & 61 & 128 & 159 & \\ 11 & 41 & 121 & 311 & 671 & 1381 & \\ \vdots & & & & & & \end{pmatrix}$$

The generating function of the sequence $a_{0,m}$ is $A_0(z) = \frac{e^z}{1+z}$. It follows from (4.1) that $B_0(z) = \exp(e^z - z - 1)$, and we notice that $a_{n,0}$ is the number v_n of partitions of $\{1, 2, \dots, n\}$ without singletons (see for instance [10]). By (3.3), one can then deduce that

$$\sum_{k=0}^m s(m, k) v_{n+k} = \sum_{k=0}^n (-1)^{m+k} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m D_{m+k}$$

If $m = 0$, we have

$$v_n = \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} D_k.$$

Example 4. The exponential generating function of the Bernoulli polynomials $B_n(x)$ is

$$\mathcal{B}_0(z) := \frac{ze^{xz}}{e^z - 1} = \sum_{n \geq 0} B_n(x) \frac{z^n}{n!}.$$

By Theorem 4, we have

$$\mathcal{A}_0(z) = \frac{(1+z)^x \ln(1+z)}{z},$$

It is not difficult to show that

$$[z^m] \mathcal{A}_0(z) = \sum_{i=0}^m (-1)^{m-i} \frac{(x)_i}{m-i+1},$$

where $[z^n] f(z)$ denote the operation of extracting the coefficient of z^n in the formal power series $f(z) = \sum f_n z^n$. Now, let us consider \mathcal{S} defined by (1.4) with the final sequence $a_{n,0} = B_n(x)$, by (3.3), we have

$$\sum_{k=0}^m s(m, k) B_{n+k}(x) = \sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m \sum_{i=0}^{m+k} (-1)^{m+k-i} \frac{(x)_i}{m+k-i+1}$$

Example 5. Catalan and Motzkin numbers naturally appear in a large number of combinatorial objects. It is well known that the Catalan number

$C_n = \frac{1}{n+1} \binom{2n}{n}$ and Motzkin number $M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{n}{2k} \binom{2k}{k}$ are connected by [1]

$$C_{n+1} = \sum_{k=0}^n \binom{n}{k} M_k \iff M_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} C_{k+1}.$$

Using the generalized Stirling transform, we can show that the Catalan numbers are related with Motzkin numbers in terms of Stirling numbers by

$$(4.3) \quad \sum_{k=0}^n s(n, k) M_k = \sum_{k=0}^{n+1} s(n+1, k) C_k,$$

and

$$(4.4) \quad C_n = \delta_{n,0} + \sum_{k=1}^n \sum_{i=0}^{k-1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} s(k-1, i) M_i \iff M_n = \sum_{k=0}^n \sum_{i=0}^{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} s(k+1, i) C_i.$$

Setting the final sequence $a_{n,0} = C_n$, we get the following matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & -5 & 29 & \cdots \\ 1 & 2 & 3 & 3 & 1 & 0 & -1 & 7 & \\ 2 & 5 & 9 & 10 & 4 & -1 & 1 & -1 & \\ 5 & 14 & 28 & 34 & 15 & -4 & 5 & -11 & \\ 14 & 42 & 90 & 117 & 56 & -15 & 19 & -42 & \\ 42 & 132 & 297 & 407 & 209 & -56 & 72 & -160 & \\ 132 & 429 & 1001 & 1430 & 780 & -208 & 272 & -614 & \\ \vdots & & & & & & & & \end{pmatrix}.$$

Since

$$\mathcal{B}_0(z) = \sum_{n \geq 0} C_n \frac{z^n}{n!} = {}_1F_1 \left(\begin{matrix} 1/2 \\ 2 \end{matrix}; 4z \right),$$

where ${}_1F_1 \left(\begin{matrix} p \\ q \end{matrix}; z \right) = \sum_{n \geq 0} \frac{\langle p \rangle_n}{\langle q \rangle_n} \frac{z^n}{n!}$. It follows from (4.2) that

$$\mathcal{A}_0(z) = \sum_{n \geq 0} R_n \frac{z^n}{n!} = {}_1F_1 \left(\begin{matrix} 1/2 \\ 2 \end{matrix}; 4 \ln(1+z) \right),$$

and

$$(4.5) \quad \sum_{k=0}^m s(m, k) C_{n+k} = \sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m R_{m+k}$$

Now, if the initial sequence $a_{0,m} = R_{m+1}$, we get the following matrix

$$\mathcal{T} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & -5 & 29 & -196 & \cdots \\ 1 & 2 & 2 & 1 & -1 & 4 & -22 & 146 & \\ 2 & 4 & 5 & 2 & 0 & -2 & 14 & -100 & \\ 4 & 9 & 12 & 6 & -2 & 4 & -16 & 93 & \\ 9 & 21 & 30 & 16 & -4 & 4 & -3 & -26 & \\ 21 & 51 & 76 & 44 & -12 & 17 & -44 & 172 & \\ 51 & 127 & 196 & 120 & -31 & 41 & -92 & 282 & \\ \vdots & & & & & & & & \end{pmatrix}.$$

From this matrix we observe that $a_{n,0} = M_n$. We prove this observation using generating functions. We have

$$\begin{aligned} A_0(z) &= \sum_{n \geq 0} R_{n+1} \frac{z^n}{n!} \\ &= \frac{d}{dz} \left(\sum_{n \geq 0} R_n \frac{z^n}{n!} \right) \\ &= \frac{1}{1+z} {}_1F_1 \left(\begin{matrix} 3/2 \\ 3 \end{matrix}; 4 \ln(1+z) \right). \end{aligned}$$

From (4.1), we get

$$\begin{aligned}
B_0(z) &= {}_1F_1\left(\frac{3}{2}; 4z\right) e^{-z} \\
&= \frac{d}{dz} \left(\sum_{n \geq 0} C_n \frac{z^n}{n!} \right) \sum_{n \geq 0} (-1)^n \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \left(\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} C_{k+1} \right) \frac{z^n}{n!} \\
&= \sum_{n \geq 0} M_n \frac{z^n}{n!}.
\end{aligned}$$

It follows

$$(4.6) \quad \sum_{k=0}^m s(m, k) M_{n+k} = \sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m R_{m+k+1}.$$

Combining results (4.5) and (4.6) gives (4.3) and (4.4).

5. HANKEL TRANSFORM

The Hankel transform of a sequence α_n is the sequence of Hankel determinants $\det(\alpha_{i+j})_{0 \leq i, j \leq n}$. A number of methods for computing the Hankel determinants have been widely investigated [4, 5, 6, 9]. It is well known that the Hankel transform of sequences α_n and β_n are equal under the binomial transform [7]

$$\beta_n = \sum_{k=0}^n \binom{n}{k} \alpha_k.$$

A natural question arises: "What about the Hankel transform of the sequences a_n and b_n under the Stirling transform?" In this section we show that there is a connection between the generalized Stirling transform and the Hankel determinants.

Theorem 5. For $n \geq 0$, we have

$$\det(a_{i,j})_{0 \leq i, j \leq n} = \det(b_{i+j})_{0 \leq i, j \leq n}.$$

Proof. We can write

$$\det(b_{i+j})_{0 \leq i, j \leq n} = \begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{n,0} \\ a_{1,0} & a_{2,0} & a_{3,0} & a_{4,0} & \cdots & a_{n+1,0} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n,0} & a_{n+1,0} & a_{n+2,0} & a_{n+3,0} & \cdots & a_{2n,0} \end{vmatrix},$$

after applying (1.4), the determinant is unchanged

$$\det (b_{i+j})_{0 \leq i, j \leq n} = \begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} - a_{1,0} & a_{3,0} - a_{2,0} - 2(a_{2,0} - a_{1,0}) & \cdots \\ a_{1,0} & a_{2,0} & a_{3,0} - a_{2,0} & a_{4,0} - a_{3,0} - 2(a_{3,0} - a_{2,0}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ a_{n,0} & a_{n+1,0} & a_{n+2,0} - a_{n+1,0} & a_{n+3,0} - a_{n+1,0} - 2(a_{n+2,0} - a_{n+1,0}) & \cdots \end{vmatrix}.$$

Using (3.2), we get

$$\det (b_{i+j})_{0 \leq i, j \leq n} = \begin{vmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & & \ddots & \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & & a_{n,n} \end{vmatrix},$$

from which the relation follows. \square

The answer to the previous question is given in the following

Corollary 2. *For $n \in \mathbb{N}_0$, we have*

$$\det (b_{i+j})_{0 \leq i, j \leq n} = \det \left(\sum_{k=0}^i \left\{ \begin{matrix} i+j \\ k+j \end{matrix} \right\}_j a_{k+j} \right)_{0 \leq i, j \leq n}.$$

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